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Testing high-energy factorization beyond the next-to-leading-logarithmic accuracy

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Abstract: By taking the high-energy limit of the two-loop and the three-loop four-point amplitudes in the maximally supersymmetric $N = 4$ Yang-Mill theory (MSYM), we test the validity of the loop expansion of the high-energy amplitude, beyond the next-to-leading-logarithmic (NLL) accuracy. We compute the three-loop Regge trajectory, and the two-loop and three-loop coefficient functions. These quantities are relevant for the BFKL evolution beyond NLL, as well as building MSYM two-loop and three-loop amplitudes with many legs in the high-energy limit, which in turn may be used as a powerful check of the evaluation of the corresponding exact amplitudes.

Keywords: 1/N Expansion, AdS-CFT Correspondence, QCD.

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Contents

1. Introduction

In the high-energy limit (HEL) $s \gg |t|$, any scattering process is dominated by the exchange of the highest-spin particle in the crossed channel. Thus, in perturbative QCD the leading contribution in powers of s/t to any scattering process comes from gluon exchange in the t channel. Building upon this fact, the BFKL theory models strong-interaction processes with two large and disparate scales, by resumming the radiative corrections to parton-parton scattering. This is achieved to leading logarithmic (LL) accuracy, in $\ln(s/|t|)$, through the BFKL equation [1 − 3], i.e. a two-dimensional integral equation which describes the evolution of the t-channel gluon propagator in transverse momentum space and moment space. The integral equation is obtained by computing the one-loop LL corrections to the gluon exchange in the t channel. They are formed by a real correction, the emission of a gluon along the ladder [4 − 6], and the leading virtual contribution of the gluon loop, the one-loop Regge trajectory [1]. The next-to-leading-logarithmic (NLL) corrections to the BFKL equation have been computed as well [7, 8]. The virtual part of the NLL kernel is provided by the two-loop trajectory [9 − 13], whose evaluation is based upon the assumption of Regge factorization of the scattering amplitudes beyond leading logarithmic accuracy.

Recently, Bern, Dixon and Smirnov (BDS) have proposed an ansatz [14] for the l-loop n-gluon scattering amplitude in the maximally supersymmetric $N = 4$ Yang-Mill theory (MSYM), with the maximally-helicity violating (MHV) configuration and for arbitrary l and n. In ref. [14], an analytic expression for the exact three-loop four-gluon amplitude, as well as iterative relations for the n-point MHV MSYM amplitudes, have also been provided. Using the results of ref. [14], we test the high-energy factorization of the gluon-gluon scattering process. The paper is organised as follows: in section 2, we analyse high-energy factorization at three-loop accuracy in perturbative QCD; in section 3, we take the exact two-loop and three-loop four-point MSYM amplitudes in the HEL, and derive the three-loop Regge trajectory, as well as the two-loop and three-loop coefficient functions; in section 4, we make contact between these quantities, the BDS ansatz and the iterative relations for the MHV MSYM amplitudes; in section 5, we sketch an outlook for the BFKL evolution in MSYM beyond NLL accuracy.

2. High-energy factorization

In perturbative QCD, the simplest process is parton-parton scattering, which in the HEL occurs through gluon exchange in the t channel. It is convenient to focus on gluon-gluon scattering. In the HEL, the tree-level amplitude for $g_a\, g_b \to g_{a'}\, g_{b'}$ may be written as [1],

$$\mathcal{M}_4^{(0)} = 2s \left[i\, g_S\, f^{aca'}\, C^{(0)}(p_a, p_{a'}) \right] \frac{1}{t} \left[i\, g_S\, f^{bcb'}\, C^{(0)}(p_b, p_{b'}) \right] , \tag{2.1}$$

where a, a', b, b' represent the colours of the scattering gluons. The gluon coefficient functions $C^{(0)}$, which yield the LO gluon impact factors, are given in ref. [1] in terms of their spin structure and in ref. [15, 16] at fixed helicities of the external gluons.

The colour decomposition of the tree-level n-gluon amplitude in a helicity basis is [17]

$$\mathcal{M}_n^{(0)} = 2^{n/2}\, g^{n-2} \sum_{S_n/Z_n} \mathrm{tr}(T^{d_{\sigma(1)}} \cdots T^{d_{\sigma(n)}})\, m_n^{(0)}(p_{\sigma(1)}, \nu_{\sigma(1)}; \ldots ; p_{\sigma(n)}, \nu_{\sigma(n)})\, , \tag{2.2}$$

where $d_1, \ldots, d_n$, and $\nu_1, \ldots, \nu_n$ are respectively the colours and the polarizations of the gluons, the T's are the colour matrices[1] in the fundamental representation of SU(N) and the sum is over the noncyclic permutations S_n/Z_n of the set $[1, \ldots, n]$. We take all the momenta as outgoing, and consider the MHV configurations $(-,-,+,\ldots,+)$ for which the tree-level gauge-invariant colour-stripped sub-amplitudes, $m_n^{(0)}(p_1, \nu_1; \ldots ; p_n, \nu_n)$, assume the form

$$m_n^{(0)}(-,-,+,\ldots,+) = \frac{\langle p_i p_j \rangle^4}{\langle p_1 p_2 \rangle \cdots \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}\, , \tag{2.3}$$

where i and j are the gluons of negative helicity. In gluon-gluon scattering, only the helicity configurations $(-,-,+,+)$ occur at tree level, and out of the six possible colour configurations only four are leading in the HEL [18]. They are those corresponding to s-channel helicity conservation. For instance, let us label the gluons clockwise and consider the helicity configuration $(b-, a-, a'+, b'+)$, with a and b incoming and a' and b' outgoing, then the four leading colour configurations are (b, a, a', b'), (b, b', a', a), (b, b', a, a'), (b, a', a, b'). The latter two, corresponding to the helicity ordering $(-,+,-,+)$, can be obtained from the former two, corresponding to the helicity ordering $(-,-,+,+)$, by $s \leftrightarrow u$ channel exchange. In addition, in the HEL $m_4^{(0)}(-,+,-,+) = -m_4^{(0)}(-,-,+,+)$, thus the different colour configurations contribute to eq. (2.2) with alternating signs, in such a way that the traces of T matrices combine to form the structure constants of eq. (2.1) [15].

The virtual radiative corrections to eq. (2.1) in LL approximation are obtained, to all orders in α_S, by replacing [1]

$$\frac{1}{t} \to \frac{1}{t} \left(\frac{s}{-t} \right)^{\alpha(t)}\, , \tag{2.4}$$

in eq. (2.1), where $\alpha(t)$ can be written in dimensional regularization in $d = 4-2\epsilon$ dimensions as

$$\alpha(t) = g_S^2\, c_\Gamma \left(\frac{\mu^2}{-t} \right)^{\epsilon} N\, \frac{2}{\epsilon}, \tag{2.5}$$

[1]We use the normalization $\mathrm{tr}(T^c T^d) = \delta^{cd}/2$, although it is immaterial in what follows.

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with N colours, and

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{2.6}$$

The fact that higher order corrections to gluon exchange in the t channel can be accounted for by dressing the gluon propagator with the exponential of eq. (2.4) is called the gluon reggeization, with $\alpha(t)$ the Regge trajectory. In order to go beyond the LL approximation, we need a prescription that disentangles the virtual corrections to the coefficient functions in eq. (2.1) from those that reggeize the gluon (2.4). The prescription for doing so is supplied by the general form of the high-energy amplitude for gluon-gluon scattering which arises from a single reggeized gluon exchanged in the crossed channel [19]

$$\mathcal{M}_4 = s \left[i\, g_S\, f^{aca'}\, C(p_a, p_{a'}) \right] \frac{1}{t} \left[\left(\frac{-s}{-t} \right)^{\alpha(t)} + \left(\frac{s}{-t} \right)^{\alpha(t)} \right] \left[i\, g_S\, f^{bcb'}\, C(p_b, p_{b'}) \right]. \tag{2.7}$$

The first Regge trajectory in eq. (2.7) corresponds to the s-channel physical region; the second to the u channel. The s and u channels have different analytic properties, however eq. (2.7) presumes that the corresponding sub-amplitudes still differ only by a sign $m_4(-,+,-,+) = -m_4(-,-,+,+)$. It is clear that the equals sign in eq. (2.7) cannot be expected to hold strictly, but, in fact, holds only up to NLL accuracy [20]. However, we show that, at the colour-stripped amplitude level, the following high-energy prescription is valid beyond NLL accuracy,

$$m_4(-,-,+,+) \equiv m_4^s = s\left[g_S\, C(p_a, p_{a'}) \right] \frac{1}{t} \left(\frac{-s}{-t} \right)^{\alpha(t)} \left[g_S\, C(p_b, p_{b'}) \right], \tag{2.8}$$

in the s-channel physical region, and similarly

$$m_4(-,+,-,+) \equiv m_4^u = s\left[g_S\, C(p_a, p_{a'}) \right] \frac{1}{t} \left(\frac{s}{-t} \right)^{\alpha(t)} \left[g_S\, C(p_b, p_{b'}) \right], \tag{2.9}$$

in the u-channel physical region. In eqs. (2.7)–(2.9), the gluon Regge trajectory has the perturbative expansion,

$$\alpha(t) = \tilde{g}_S^2(t)\alpha^{(1)} + \tilde{g}_S^4(t)\alpha^{(2)} + \tilde{g}_S^6(t)\alpha^{(3)} + \mathcal{O}(\tilde{g}_S^8)\,, \tag{2.10}$$

with the rescaled coupling

$$\tilde{g}_S^2(t) = g_S^2 c_\Gamma \left(\frac{\mu^2}{-t} \right)^{\epsilon}, \tag{2.11}$$

and with $\alpha^{(1)} = 2N/\epsilon$ given in eq. (2.5). The coefficient functions C can be written as

$$C = C^{(0)}(1 + \tilde{g}_S^2(t)C^{(1)} + \tilde{g}_S^4(t)C^{(2)} + \tilde{g}_S^6(t)C^{(3)}) + \mathcal{O}(\tilde{g}_S^8)\,. \tag{2.12}$$

Because the coefficient function C is real (up to overall complex phases in $C^{(0)}$ induced by the complex-valued helicity bases), eqs. (2.7)–(2.9) imply that the imaginary part of the amplitude comes entirely from the Regge trajectory in the s channel, according to the usual prescription $\ln(-s) = \ln(s) - i\pi$, for $s > 0$.

The expansion of eqs. (2.8) and (2.9) can be written as,

$$m_4^i = m_4^{i(0)} \left(1 + \tilde{g}_S^2\, m_4^{i(1)} + \tilde{g}_S^4 m_4^{i(2)} + \tilde{g}_S^6 m_4^{i(3)} + \mathcal{O}(\tilde{g}_S^8)\right) , \tag{2.13}$$

with $i = s, u$. The one-loop coefficients of eq. (2.13) are,

$$\begin{aligned} m_4^{u(1)} &= \alpha^{(1)} L + \ 2 C^{(1)} , \\ m_4^{s(1)} &= m_4^{u(1)} - i\pi\alpha^{(1)} . \end{aligned} \tag{2.14}$$

with $L = \ln\left(\frac{s}{-t}\right)$. The one-loop trajectory, $\alpha^{(1)}$, is universal, i.e. it is independent of the type of parton undergoing the high-energy scattering process. It is also independent of the infrared (IR) regularisation scheme. Conversely, the one-loop coefficient function, $C^{(1)}$, is process and IR-scheme dependent. $C^{(1)}$ was computed in conventional dimensional regularization (CDR)/'t-Hooft-Veltman (HV) schemes in ref. [19 − 23], and in the dimensional reduction scheme (DRED) in ref. [20, 23].

The two-loop coefficients of eq. (2.13) are,

$$\begin{aligned} m_4^{u(2)} &= \frac{1}{2}\left(\alpha^{(1)}\right)^2 L^2 + \left(\alpha^{(2)} + 2\, C^{(1)}\alpha^{(1)}\right)\ L + 2\, C^{(2)} + \left(C^{(1)}\right)^2 , \\ m_4^{s(2)} &= m_4^{u(2)} - \frac{\pi^2}{2}\left(\alpha^{(1)}\right)^2 - i\pi\left[\left(\alpha^{(1)}\right)^2 L + \alpha^{(2)} + 2\, C^{(1)}\alpha^{(1)}\right] . \end{aligned} \tag{2.15}$$

The two-loop trajectory, $\alpha^{(2)}$, was computed in the CDR scheme in ref. [9 − 13]. We note that the coefficients of the double and the real parts of the single logarithms are the same in the s and the u channel. Therefore, it is correct to use eq. (2.7) at NLL accuracy. However, the real part of the constant term is channel dependent, and thus care must be used in extracting the two-loop coefficient function from it.[2]

The three-loop coefficients of eq. (2.13) are,

$$\begin{aligned} m_4^{u(3)} &= \frac{1}{3!}\left(\alpha^{(1)}\right)^3 L^3 + \alpha^{(1)}\left(\alpha^{(2)} + C^{(1)}\alpha^{(1)}\right) L^2 \\ &\quad + \left[\alpha^{(3)} + 2\,\alpha^{(2)} C^{(1)} + \alpha^{(1)}\left(2\, C^{(2)} + \left(C^{(1)}\right)^2\right)\right] L + 2\, C^{(3)} + 2\, C^{(2)} C^{(1)} , \end{aligned} \tag{2.16}$$

$$\begin{aligned} m_4^{s(3)} &= m_4^{u(3)} - \frac{\pi^2}{2}\left(\alpha^{(1)}\right)^3\ L - \pi^2\alpha^{(1)}\left(\alpha^{(2)} + C^{(1)}\alpha^{(1)}\right) \\ &\quad - i\pi\left[\frac{\left(\alpha^{(1)}\right)^3}{2} L^2 + 2\,\alpha^{(1)}\left(\alpha^{(2)} + C^{(1)}\alpha^{(1)}\right)\ L \right. \\ &\quad \left. + \alpha^{(3)} + 2\,\alpha^{(2)} C^{(1)} + \alpha^{(1)}\left(2\, C^{(2)} + \left(C^{(1)}\right)^2\right) - \frac{\pi^2}{3!}\left(\alpha^{(1)}\right)^3\right] . \end{aligned} \tag{2.17}$$

We note that the coefficients of the triple and the real part of the double logarithms are the same in both s and u channels, thus maintaining the correctness of eq. (2.7) at NLL

[2] In ref. [13], the constant terms for the gluon-gluon, quark-quark and gluon-quark scattering processes were evaluated by projecting the two-loop amplitude on the tree amplitude, which entails averaging over the s and u channels. However, an apparent discrepancy in the factorization between the quark and the gluon amplitudes prevented any conclusions from being drawn about the two-loop coefficient functions.

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accuracy. However, although the three-loop trajectory is universal, the coefficient of the single logarithm, and thus the extraction of the three-loop trajectory, depends on the channel under consideration.

3. The MSYM amplitudes

The QCD four-point parton-parton scattering amplitudes are known at two-loop accuracy [24−32]. In QCD, the gluon loop can be decomposed into a MSYM multiplet, an $N = 1$ chiral multiplet, and a complex scalar. The MSYM contribution is the simplest to evaluate, and captures most of the IR behaviour of the full gluon loop. In fact within MSYM, the gluon-gluon scattering amplitude is known at three loops analytically [14] and at four loops through $\mathcal{O}(1/\epsilon)$ numerically [33]. Bern, Dixon and Smirnov have proposed an ansatz [14] for the MHV l-loop n-gluon scattering amplitude for arbitrary l and n. The ansatz agrees with the direct evaluation of the three-loop four-point amplitude.[3] In addition, in ref. [40, 41] it has been shown that in the HEL the BDS ansatz for the four-point amplitude exhibits the Regge behaviour of eqs. (2.8) and (2.9). Thus, we shall use the direct evaluation of the three-loop amplitude and the BDS ansatz to derive the relevant quantities for the three-loop high-energy factorization in MSYM.

We start with the HEL of the QCD one-loop colour-stripped amplitude for gluon-gluon scattering, which is known to all orders in ϵ [23], and use the maximal trascendentality principle [42] to select the MSYM contribution. Using the conventions of eq. (2.13) and rescaling the coupling (2.11) as

$$\bar{g}_S^2(t) = \tilde{g}_S^2(t) N \,, \tag{3.1}$$

we obtain

$$\begin{aligned} m_{4_{\rm MSYM}}^{u(1)} &= 2\left(\frac{\psi(1+\epsilon) - 2\psi(-\epsilon) + \psi(1)}{\epsilon} + \frac{L}{\epsilon}\right) \qquad (3.2)\\ &= -\frac{4}{\epsilon^2} + \frac{2L}{\epsilon} + \pi^2 + 2\zeta_3\epsilon + \frac{\pi^4}{15}\epsilon^2 + 2\zeta_5\epsilon^3 + \frac{2\pi^6}{315}\epsilon^4 + \mathcal{O}(\epsilon^5) \,, \\ m_{4_{\rm MSYM}}^{s(1)} &= m_{4_{\rm MSYM}}^{u(1)} - \frac{2i\pi}{\epsilon} \,. \qquad (3.3) \end{aligned}$$

From eq. (3.2), we see that the MSYM one-loop trajectory is precisely the same as that in QCD, eq. (2.5), while the MSYM one-loop coefficient function is, to all orders in ϵ, given by,

$$C_{\rm MSYM}^{(1)} = \frac{\psi(1+\epsilon) - 2\psi(-\epsilon) + \psi(1)}{\epsilon} \,. \tag{3.4}$$

[3] The BDS ansatz was first predicted to fail by Alday and Maldacena [34], for amplitudes with a large number of gluons in the strong-coupling limit. They claimed that the finite pieces of the two-loop six gluon amplitude would be incorrectly determined. This prediction was backed up by Drummond et al. [35], who considered the finite contribution in the dual theory [36] by computing the hexagonal light-like Wilson loop at two loops. The conclusion was that either the BDS ansatz is wrong, or the equivalence between Wilson loops and scattering amplitudes does not work at two loops. Recent numerical results for the finite part of the MHV six-gluon amplitude in MSYM [37] have confirmed the equivalence with the finite part of the light-like hexagon Wilson loop [38] thereby disproving the BDS ansatz. Furthermore, the analytic structure of the two-loop six-gluon amplitude in the multi-Regge kinematics is expected to be at odds with the structure of the BDS ansatz [39].

At two loops, we take the direct calculation of the exact MSYM amplitude, given in ref. [14] to $\mathcal{O}(\epsilon^2)$, and evaluate the HEL in the u and s physical channels,

$$\begin{aligned} m^{u(2)}_{4_{\rm MSYM}} &= \frac{8}{\epsilon^4} - \frac{8L}{\epsilon^3} + \left(2L^2 - \frac{11\pi^2}{3}\right)\frac{1}{\epsilon^2} + \left(\frac{5\pi^2}{3}L - 6\zeta_3\right)\frac{1}{\epsilon} \\ &\quad + \left(2\zeta_3 L - \frac{17}{90}\pi^4\right) + \left(\frac{2\pi^4}{45}L + \frac{4\pi^2}{3}\zeta_3 - 86\zeta_5\right)\epsilon \\ &\quad + \left[\left(6\pi^2\zeta_3 + 86\zeta_5\right)L - 94\zeta_3^2 - \frac{77\pi^6}{180}\right]\epsilon^2 + \mathcal{O}(\epsilon^3)\,, \end{aligned} \tag{3.5}$$

$$\begin{aligned} m^{s(2)}_{4_{\rm MSYM}} &= m^{u(2)}_{4_{\rm MSYM}} + \frac{8i\pi}{\epsilon^3} - \frac{2\pi^2 + 4i\pi L}{\epsilon^2} - \frac{5i\pi^3}{3\epsilon} - 2\zeta_3 i\pi \\ &\quad - \frac{2i\pi^5}{45}\epsilon - \left(6\zeta_3\pi^2 + 86\zeta_5\right)i\pi\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{3.6}$$

As expected from eqs. (2.15) and (2.15), the real parts of $m^{u(2)}_{4_{\rm MSYM}}$ and $m^{s(2)}_{4_{\rm MSYM}}$ differ by the constant term $2\pi^2/\epsilon^2$. Note that the average over the s and u channels is in agreement, to $\mathcal{O}(\epsilon^0)$, with the terms of highest transcendentality of the HEL of the projection of the two-loop amplitude on the tree amplitude [13].

Equating the coefficients of the single logarithm of eqs. (2.15) and (3.5), we obtain the MSYM two-loop trajectory [42] in the DRED scheme,

$$\alpha^{(2)} = -\frac{\pi^2}{3\epsilon} - 2\zeta_3 - \frac{4\pi^4}{45}\epsilon + (6\pi^2\zeta_3 + 82\zeta_5)\epsilon^2 + \mathcal{O}(\epsilon^3)\,, \tag{3.7}$$

which is in agreement, to $\mathcal{O}(\epsilon^0)$, with the terms of highest transcendentality of the QCD two-loop trajectory [13].

Equating the coefficients of the constant term of eqs. (2.15) and (3.5), we obtain the MSYM two-loop coefficient function,

$$\begin{aligned} C^{(2)}_{\rm MSYM} &= \frac{2}{\epsilon^4} - \frac{5\pi^2}{6}\frac{1}{\epsilon^2} - \frac{\zeta_3}{\epsilon} - \frac{11}{72}\pi^4 \\ &\quad + \left(\frac{\pi^2}{6}\zeta_3 - 41\zeta_5\right)\epsilon - \left(\frac{95}{2}\zeta_3^2 + \frac{113\pi^6}{504}\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{3.8}$$

The evaluation of eq. (3.8) has been performed also in the s channel through eqs. (2.15) and (3.6), thus checking the consistency of eqs. (2.8) and (2.9) at two loops.

At three loops, we take the direct calculation of the exact MSYM amplitude, provided in ref. [14] to $\mathcal{O}(\epsilon^0)$, and evaluate the HEL in the u and s physical channels. We find

$$\begin{aligned} m^{u(3)}_{4_{\rm MSYM}} &= -\frac{32}{3\epsilon^6} + \frac{16L}{\epsilon^5} - \left(8L^2 - \frac{20\pi^2}{3}\right)\frac{1}{\epsilon^4} + \left[\frac{4}{3}L^3 - 6\pi^2 L + 8\zeta_3\right]\frac{1}{\epsilon^3} \\ &\quad + \left[\frac{4\pi^2}{3}L^2 - 4\zeta_3 L + \frac{181}{405}\pi^4\right]\frac{1}{\epsilon^2} - \left(\frac{26}{135}\pi^4 L + \frac{112}{27}\zeta_3\pi^2 - \frac{952}{3}\zeta_5\right)\frac{1}{\epsilon} \\ &\quad - \frac{2\pi^4}{45}L^2 - \left(484\zeta_5 + \frac{196\pi^2}{9}\zeta_3\right)L + \frac{3284}{9}\zeta_3^2 + \frac{88747\pi^6}{51030} + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{3.9}$$

$$\begin{aligned} m^{s(3)}_{4_{\rm MSYM}} &= m^{u(3)}_{4_{\rm MSYM}} - \frac{16i\pi}{\epsilon^5} + \frac{8\pi^2+16Li\pi}{\epsilon^4} - \left[4\pi^2 L + \left(4L^2 - \frac{22\pi^2}{3}\right) i\pi\right]\frac{1}{\epsilon^3} \\ &\quad - \left[\frac{4\pi^4}{3} + \left(\frac{8\pi^2}{3}L - 4\zeta_3\right) i\pi\right]\frac{1}{\epsilon^2} + \frac{26}{135}\frac{i\pi^5}{\epsilon} \\ &\quad + \frac{2\pi^6}{45} + \left(\frac{4\pi^4}{45}L + \frac{196}{9}\zeta_3\pi^2 + 484\zeta_5\right)\pi i + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{3.10}$$

The coefficients of the double logarithm of eqs. (2.16) and (3.9) provide a cross check of the two-loop trajectory $\alpha^{(2)}$, while comparing the coefficients of the single logarithm we obtain the MSYM three-loop trajectory,

$$\alpha^{(3)} = \frac{22}{135}\frac{\pi^4}{\epsilon} + \frac{20\pi^2}{9}\zeta_3 + 16\zeta_5 + \mathcal{O}(\epsilon)\,, \tag{3.11}$$

in agreement with ref. [39, 40]. The trajectory was also implicitly evaluated in ref. [41]. Equating the coefficients of the constant term of eqs. (2.16) and (3.9) we find the MSYM three-loop coefficient function,

$$\begin{aligned} C^{(3)}_{\rm MSYM} &= -\frac{4}{3\epsilon^6} + \frac{2\pi^2}{3}\frac{1}{\epsilon^4} + \frac{217\pi^4}{810}\frac{1}{\epsilon^2} + \left(-\frac{11\pi^2}{27}\zeta_3 + \frac{224}{3}\zeta_5\right)\frac{1}{\epsilon} \\ &\quad + \left(\frac{796}{9}\zeta_3^2 + \frac{211861\pi^6}{408240}\right) + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{3.12}$$

The evaluation of eq. (3.12) has been performed also in the s channel through eqs. (2.17) and (3.10), thus checking the consistency of eqs. (2.8) and (2.9) at three loops.

4. The Bern-Dixon-Smirnov ansatz

The BDS ansatz prescribes that the n-point MHV amplitude be written as,

$$\begin{aligned} m_n &= m_n^{(0)}\left[1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon)\right] \\ &= m_n^{(0)} \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon)\right)\right]\,, \end{aligned} \tag{4.1}$$

where

$$a = \frac{2g_S^2 N}{(4\pi)^{2-\epsilon}} e^{-\gamma\epsilon} \tag{4.2}$$

is the 't-Hooft gauge coupling, and with

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}\,, \tag{4.3}$$

where $f^{(1)}(\epsilon) = 1$, and $f_0^{(l)}$ is proportional to the l-loop cusp anomalous dimension, $\hat\gamma_K^{(l)} = 4f_0^{(l)}$ and $f_1^{(l)}$ is related to another quantity, $\mathcal{G}_0^{(l)} = 2f_1^{(l)}/l$, which enters the IR Sudakov form factor and accounts for virtual divergences which are not simultaneously soft and collinear [43, 44]. In eq. (4.1), $Const^{(l)}$ are constants, and $E_n^{(l)}(\epsilon)$ are $\mathcal{O}(\epsilon)$ contributions, with $Const^{(1)} = 0$ and $E_n^{(1)}(\epsilon) = 0$, and $M_n^{(L)}(\epsilon)$ is the L-loop colour-stripped amplitude rescaled by the tree amplitude. In the convention and notation of eq. (2.13), the four-point amplitude is given by,

$$a^L M_4^{(L)}(\epsilon) = \left(\frac{a}{2G(\epsilon)}\right)^L \left(\frac{\mu^2}{-t}\right)^{L\epsilon} m_4^{(L)}(\epsilon)\,, \tag{4.4}$$

with

$$G(\epsilon) = \frac{e^{-\gamma\epsilon}\,\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)} = 1 + \mathcal{O}(\epsilon^2)\,, \tag{4.5}$$

Thus, using the rescaled coupling (3.1), the BDS ansatz (4.1) for the four-point amplitude becomes

$$\begin{aligned} m_4 &= m_4^{(0)} \left[1 + \sum_{L=1}^{\infty} \bar{g}_S^{2L}(t) m_4^{(L)}(\epsilon)\right] \\ &= m_4^{(0)} \exp\left[\sum_{l=1}^{\infty} \bar{g}_S^{2l}(t)\,(2G(\epsilon))^l \left(f^{(l)}(\epsilon)\frac{m_4^{(1)}(l\epsilon)}{2G(l\epsilon)} + Const^{(l)} + \mathcal{O}(\epsilon)\right)\right]. \end{aligned} \tag{4.6}$$

eq. (4.6) applies equally well in either the s or the u channels. Substituting the HEL one-loop amplitude (2.14) in eq. (4.6) and comparing with the expansion (2.13) of the high-energy factorization (2.8) or (2.9), we see that the coefficient of the single logarithm allows us to read off the value of the gluon trajectory,

$$\begin{aligned} \alpha^{(1)} &= \frac{2}{\epsilon}\, G(\epsilon) \\ \alpha^{(2)} &= \frac{2}{\epsilon}\, G^2(\epsilon) f^{(2)}(\epsilon) \\ \alpha^{(3)} &= \frac{8}{3\epsilon}\, G^3(\epsilon) f^{(3)}(\epsilon)\,, \end{aligned} \tag{4.7}$$

and in general

$$\alpha^{(l)} = \frac{2^l}{l\epsilon}\, G^l(\epsilon) f^{(l)}(\epsilon)\,. \tag{4.8}$$

From eq. (4.8), we see that only the first two terms of the $f^{(l)}(\epsilon)$ function (4.3) enter the evaluation of the Regge trajectory. Dropping $G(\epsilon)$, which does not contribute in eq. (4.7) to $\mathcal{O}(\epsilon^0)$, and using the $f^{(2)}$ and $f^{(3)}$ functions [14],

$$\begin{aligned} f^{(2)}(\epsilon) &= -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2\,, \\ f^{(3)}(\epsilon) &= \frac{11}{2}\zeta_4 + (6\zeta_5 + 5\zeta_2\zeta_3)\epsilon + (c_1\zeta_6 + c_2\zeta_3^2)\epsilon^2\,, \end{aligned} \tag{4.9}$$

we see that eq. (4.7) agrees with eqs. (3.7) and (3.11) to $\mathcal{O}(\epsilon^0)$. The coefficients c_1, c_2 are unknown, but they do not enter the evaluation of the Regge trajectory.

Through the iterative structure of the MSYM amplitudes, it is possible to express the coefficient functions at a given loop in terms of the coefficient functions at a lower number of loops. Using eq. (4.6), the iterative structure of the two-loop four-point MSYM amplitude [45, 46] is given by

$$m_4^{(2)}(\epsilon) = \frac{1}{2}\left[m_4^{(1)}(\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, m_4^{(1)}(2\epsilon) + 4\,Const^{(2)} + \mathcal{O}(\epsilon)\,, \tag{4.10}$$

with $Const^{(2)} = -\zeta_2^2/2$, and where the one-loop amplitude must be known to $\mathcal{O}(\epsilon^2)$. Using the two-loop factorization, in either the u (2.15) or s channels (2.15), we find

$$C^{(2)}_{\rm MSYM}(\epsilon) = \frac{1}{2}\left[C^{(1)}_{\rm MSYM}(\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, C^{(1)}_{\rm MSYM}(2\epsilon) + 2\,Const^{(2)} + \mathcal{O}(\epsilon)\,, \tag{4.11}$$

where the one-loop coefficient function is needed to $\mathcal{O}(\epsilon^2)$. eq. (4.11) agrees with eq. (3.8) to $\mathcal{O}(\epsilon^0)$.

Similarly, the iterative structure of the three-loop four-point MSYM amplitude is [14],

$$m_4^{(3)}(\epsilon) = -\frac{1}{3}\left[m_4^{(1)}(\epsilon)\right]^3 + m_4^{(2)}(\epsilon)\, m_4^{(1)}(\epsilon) + \frac{4\,G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, m_4^{(1)}(3\epsilon) + 8\,Const^{(3)} + \mathcal{O}(\epsilon)\,, \tag{4.12}$$

where $m_4^{(1)}(\epsilon)$ and $m_4^{(2)}(\epsilon)$ must be known to $\mathcal{O}(\epsilon^4)$ and $\mathcal{O}(\epsilon^2)$, respectively, and with

$$Const^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2\,. \tag{4.13}$$

Using the three-loop factorization, in either the u (2.16) or s channels (2.17), we obtain the three-loop coefficient function

$$\begin{aligned} C^{(3)}_{\rm MSYM}(\epsilon) = &-\frac{1}{3}\left[C^{(1)}_{\rm MSYM}(\epsilon)\right]^3 + C^{(1)}_{\rm MSYM}(\epsilon)\, C^{(2)}_{\rm MSYM}(\epsilon) \\ &+ \frac{4\,G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, C^{(1)}_{\rm MSYM}(3\epsilon) + 4\,Const^{(3)} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{4.14}$$

The coefficients c_1, c_2 cancel when eqs. (4.9) and (4.13) are used in eqs. (4.12) and (4.14). Using the two-loop coefficient function to $\mathcal{O}(\epsilon^2)$ (3.8), and the one-loop coefficient function to $\mathcal{O}(\epsilon^4)$ (3.4), we see that eq. (4.14) is in agreement with eq. (3.12) to $\mathcal{O}(\epsilon^0)$.

5. Conclusions

The iterative structure and the exponentiated form of the MSYM amplitudes in the MHV configuration offer a useful computational lab to test high-energy factorization in the MSYM, and allow the derivation of some of the relevant quantities in the high energy limit. Using the MSYM two- and three-loop four-point amplitudes, we have tested the high-energy factorization of the colour-stripped amplitude. In particular, we have shown that it is valid beyond NLL accuracy, and we have verified that the factorization formulae (2.8) and (2.9) hold at three-loop accuracy. Accordingly, we have derived the three-loop Regge trajectory (3.11), as well as the two-loop (3.8) and three-loop coefficient functions (3.12).

The three-loop Regge trajectory is one of the building blocks for deriving a BFKL evolution equation at next-to-next-to-leading logarithmic (NNLL) accuracy. The others are the tree vertex for the emission of three gluons along the ladder [47, 48], the two-loop vertex for the emission of a gluon along the ladder, and the one-loop vertex for the emission of two gluons along the ladder. In addition, if one wants to compute jet cross sections at a matching accuracy, the NNLO impact factors must be determined. The ingredients for that are the two-loop coefficient function evaluated here, as well as the tree coefficient function for the emission of three gluons [47, 48], and the one-loop coefficient function for the emission of two gluons.

Furthermore the two-loop and three-loop Regge trajectories and coefficient functions, together with the vertices for the emission of one or more gluons along the ladder which were not analysed in this work, can be used to build MSYM two-loop and three-loop amplitudes with a larger number of legs in the high-energy limit. This may serve as a powerful check on the structure of high multiplicity MSYM amplitudes.

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